Lebanese American University — Fall 2011

Byblos

Calculus III — Date: Nov/10

Test #1 — Duration: 1h 30

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ID:

I. Find the integrals

$$\int \frac{dx}{x^6-1}, \quad \int \frac{t}{(t-1)^{10}}\,dt, \quad \int \frac{u^5}{\sqrt{u^3+3}}\,du$$

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$$\frac{1}{x^6-1} = \frac{x-2}{6(x^2-x+1)} - \frac{1}{6(x+1)} + \frac{1}{6(x-1)} - \frac{x+2}{6(x^2+x+1)}$$

$$\int \frac{dx}{x^6-1} = -\frac{1}{12}\ln(x^2+x+1) - \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{\sqrt{3}}{3}(2x+1)\right) + \frac{1}{6}\ln(x-1)$$
$$+\frac{1}{12}\ln(x^2-x+1) - \frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{\sqrt{3}}{3}(2x-1)\right) - \frac{1}{6}\ln(x+1) + C$$

$$* \int \frac{t}{(t-1)^{10}}\,dt = \int \frac{t-1+1}{(t-1)^{10}}\,dt = \int \frac{dt}{(t-1)^9} + \int \frac{dt}{(t-1)^{10}}$$

$$= \int (t-1)^{-9}\,dt + \int (t-1)^{-10}\,dt = \frac{1}{-9+1}(t-1)^{-9+1} + \frac{1}{-10+1}(t-1)^{-10+1} + C$$
$$= -\frac{1}{8(t-1)^8} - \frac{1}{9(t-1)^9} + C$$

$$* \int \frac{u^5}{\sqrt{u^3+3}}\,du = \int \frac{u^5+3u^2-3u^2}{\sqrt{u^3+3}}\,du = \int u^2\sqrt{u^3+3}\,du - \int \frac{3u^2\,du}{\sqrt{u^3+3}}$$

$$= \frac{1}{3}\int 3u^2(u^3+3)^{\frac{1}{2}}\,du - 2\int \frac{3u^2\,du}{2\sqrt{u^3+3}} = \frac{1}{3}\,(u^3+3)^{\frac{3}{2}} - 2(u^3+3)^{\frac{1}{2}} + C$$
$$= \frac{1}{3}\sqrt{u^3+3}\,(u^3+3-6) = (u^3-3)\sqrt{u^3+3}$$

$$\frac{1}{3}\int 3u^2\sqrt{u^3+3}\,du - 2\int\frac{3u^2\,du}{2\sqrt{u^3+3}}$$

$$\frac{1}{3}\,\frac{1}{\frac{1}{2}+1}\left(u^3+3\right)^{\frac{3}{2}} - 2\left(u^3+3\right)^{\frac{1}{2}} =$$

$$\frac{2}{9}\left(u^3+3\right)^{\frac{3}{2}} - 2\left(u^3+3\right)^{\frac{1}{2}} = \frac{2}{9}\sqrt{u^3+3}\left(u^3-6\right) + C$$

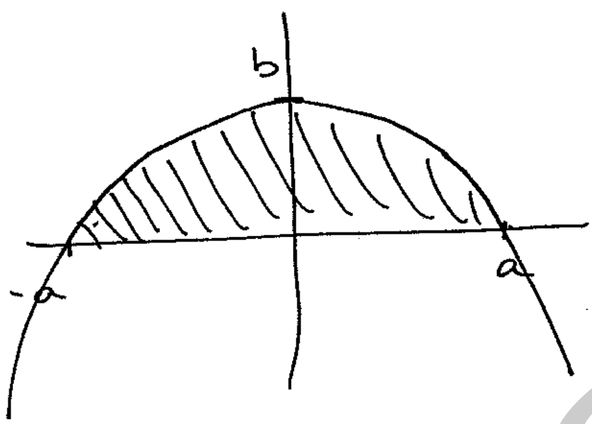
II. Let $\mathcal{P}$ be the parabola of equation $y = b\left(1 - \frac{x^2}{a^2}\right)$ with $a > 0$ and $b > 0$.

5 (a) Sketch the graphic representation of $\mathcal{P}$.

5 (b) Show that the surface $\mathcal{S}$, delimited by the positive part of $\mathcal{P}$ and the x-axis is equal to $\frac{4ab}{3}$.

10 (c) Find the volume of the solid which is the result of the rotation of $\mathcal{S}$ with respect to y-axis in function of a and b.

(a)



b) $\int_{-a}^{a} b\left(1-\frac{x^2}{a^2}\right)dx = 2\int_{0}^{a} b\left(1-\frac{x^2}{a^2}\right)$

$= \left[2bx - \frac{2b}{3}\frac{x^3}{a^2}\right]_0^a$

$= 2ba - \frac{2ba}{3} = \frac{4ab}{3}.$

c) $\int_0^a 2\pi(x)\, b\left(1-\frac{x^2}{a^2}\right)dx = 2\pi b\int_0^a \left(x - \frac{x^3}{a^2}\right)dx$

$= 2\pi b\left[\frac{x^2}{2} - \frac{x^4}{4a^2}\right]_0^a = 2\pi b\left(\frac{a^2}{2} - \frac{a^2}{4}\right) = \frac{\pi a^2 b}{2}$

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III. Let a and b be two real numbers.

5 (a) Prove the identity

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}.$$

5 (b) Deduce that

$$\tan^{-1}(a) + \tan^{-1}\left(\frac{1}{a}\right) = \frac{\pi}{2} \quad \forall a > 0.$$

5 (c) Use (b) and the substitution $u = \dfrac{1}{x}$ in order to find the integral

$$\int_{\frac{1}{2}}^{2} \left(1+\frac{1}{x^2}\right)\tan^{-1}(x)\,dx.$$

a) $\tan(a+b) = \dfrac{\sin(a+b)}{\cos(a+b)} = \dfrac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\cos(a)\cos(b)-\sin(a)\sin(b)} =$

$\dfrac{\cos(a)\cos(b)}{\cos(a)\cos(b)} \cdot \dfrac{\frac{\sin(a)}{\cos(a)}+\frac{\sin(b)}{\cos(b)}}{1-\frac{\sin(a)\sin(b)}{\cos(a)\cos(b)}} = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$

b) $\tan\left(\tan^{-1}(a) + \tan^{-1}\left(\frac{1}{a}\right)\right) = \dfrac{a+\frac{1}{a}}{\underbrace{1-a\cdot\frac{1}{a}}_{=0}} \to \infty$

$\tan\left(\frac{\pi}{2}\right) \to \infty$

$\blacksquare$

c) $\displaystyle\int_{1/2}^{2}\left(1+\frac{1}{x^2}\right)\tan^{-1}(x)\,dx + \int_{1/2}^{2}\left(1+\frac{1}{x^2}\right)\tan^{-1}\left(\frac{1}{x}\right)dx = \int_{1/2}^{2}\frac{\pi}{2}\left(1+\frac{1}{x^2}\right)dx = \frac{3\pi}{2}$

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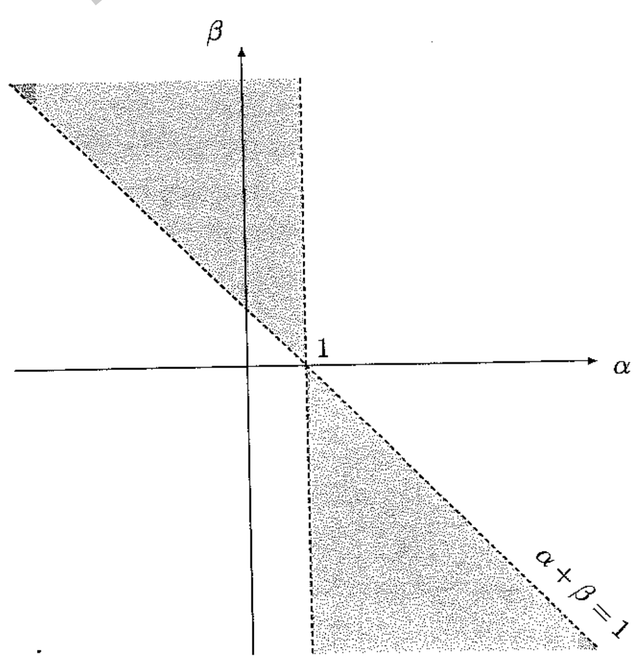
IV. Let $I = \displaystyle\int_0^{\infty} \frac{dx}{x^{\alpha}(1+x^{\beta})}$

15 (a) Find the values of the couple $(\alpha, \beta) \in \mathbb{R}$ for which I is convergent. (*Hint: Split the integration interval* $\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$).

16 (b) Sketch in the plan the set S of couples (α, β) for which I is convergent.

10 (c) Deduce, from above, the values of the couple (α, β) for which $\displaystyle\int_0^{\infty} \frac{t^{\alpha+\beta-2}}{t^{\beta}+1}\, dt$ converges.

	$\sim f(x)$ 0	$\sim f(x)$ $+\infty$	convergence of $\int_0^1 f(x)\,dx$	convergence of $\int_1^{\infty} f(x)\,dx$
$\beta > 0$	$\frac{1}{x^{\alpha}}$	$\frac{1}{x^{\alpha+\beta}}$	$\alpha < 1$	$\alpha + \beta > 1$
$\beta = 0$	$\frac{1}{2x^{\alpha}}$	$\frac{1}{2x^{\alpha}}$	$\alpha < 1$	$\alpha > 1$
$\beta < 0$	$\frac{1}{x^{\alpha+\beta}}$	$\frac{1}{x^{\alpha}}$	$\alpha + \beta < 1$	$\alpha > 1$



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V. *BONUS QUESTION* Let $\varepsilon > 0$. Find, as a function of ε, the following integral

$$I = \int_{-\varepsilon}^{\varepsilon} \frac{\tan(\frac{x}{x^2+1})\cos(x)}{(x^2+1)\sqrt{x^2+2}}\,dx$$

$\dfrac{\tan\left(\frac{x}{x^2+1}\right)\cos(x)}{(x^2+1)\sqrt{x^2+2}}$ is odd $\Rightarrow I = 0$

MARKS : I. [30] **II.** [20] **III.** [15] **IV.** [35] **V.** [10]